AM207 Lecture 2

https://am207.info/

AM207 Class Infrastructure

- Website am207.info
- Join Piazza
- Join Slack
- We may add Twitter if we're feeling adventurous so stay posted

AM207 Slack

- Please use for asking questions during lecture and lab (if you're not present to raise your hand and ask)
- The channel for the current lecture is #lecture
- The channel for the current lab is #lab
- We'll rename after class/lab to #lectureN and #labM
- Don't abuse (we'll announce any other future appropriate uses on Piazza)

Advice from your TFs

- **Collaboration** -- if you collaborate for assignments (HW and Paper/Tutorial) for which we allow students to work together PLEASE PLEASE SUBMIT ONE ASSIGNMENT.
- **Contacting Teaching Staff*** -- We pride ourselves on being available. Please come to OH (the class will be a lot easier if you do so).
- You can also email us at am207.info. Right now we have aliases for grading (grading@) and info (info@) .

Random Variables

Definition. A random variable is a mapping

$$X : \Omega \to \mathbb{R}$$

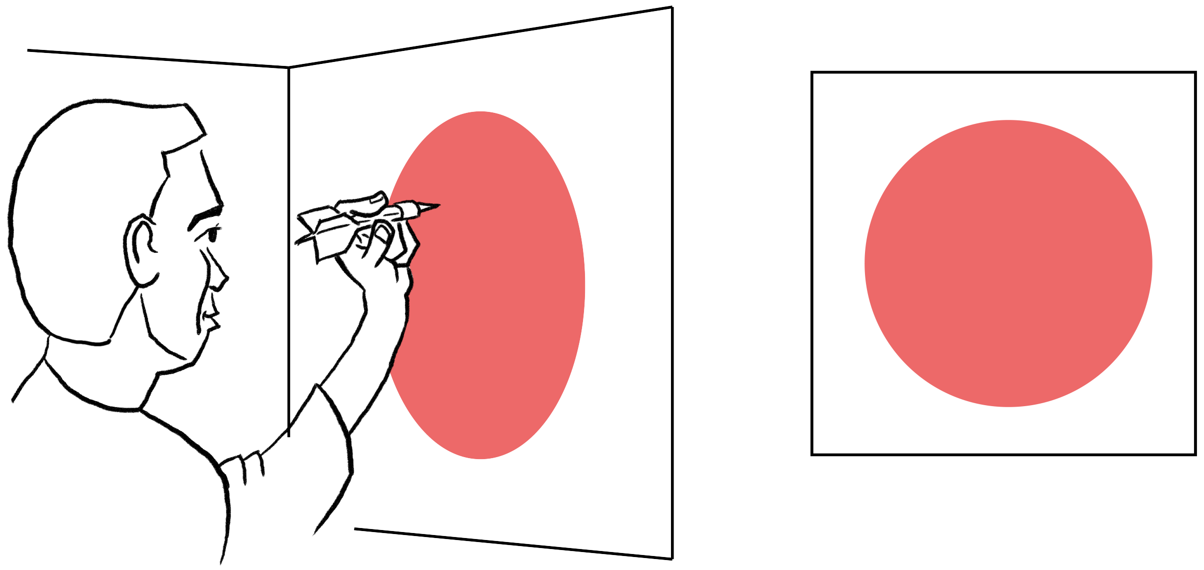
that assigns a real number $X(\omega)$ to each outcome ω.
- Ω is the sample space. Points
- ω in Ω are called sample outcomes, realizations, or elements.
- Subsets of Ω are called Events.

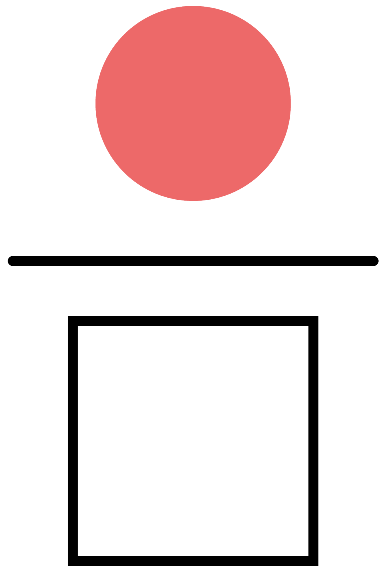
Fundamental rules of probability:

1. $p(X) >= 0$; probability must be non-negative
2. $0 \leq p(X) \leq 1$
3. $p(X) + p(X^{-}) = 1$ either happen or not happen.
4. $p(X + Y) = p(X) + p(Y) - p(X, Y)$

- Say $\omega = HHTTTTTHTT$ then $X(\omega) = 3$ if defined as number of heads in the sequence ω.
- We will assign a real number P(A) to every event A, called the probability of A.
- We also call P a probability distribution or a probability measure.

Probability as frequency



$P(A) =$ 

A Murder Mystery

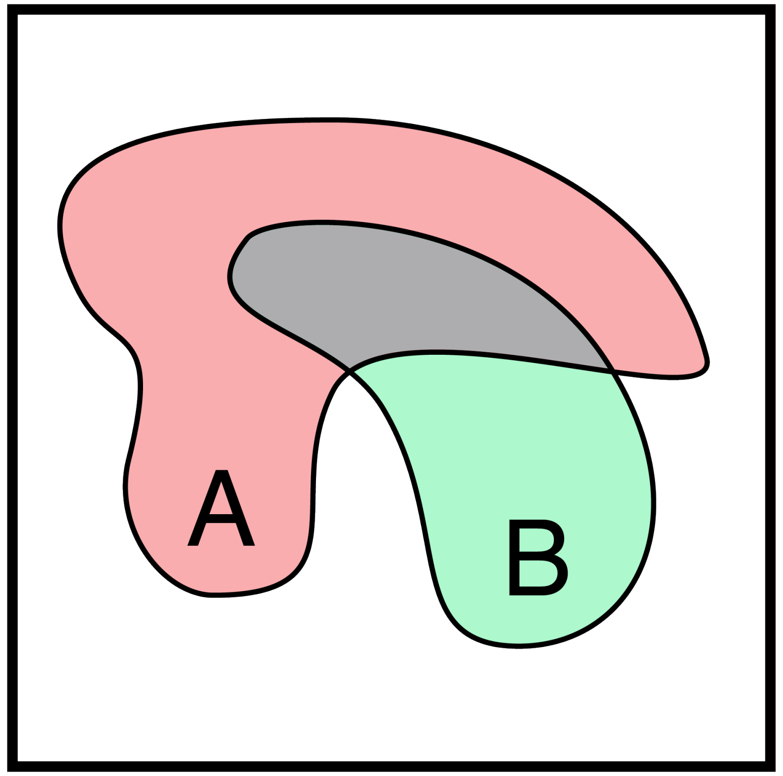
(from the book: Model Based Machine Learning)

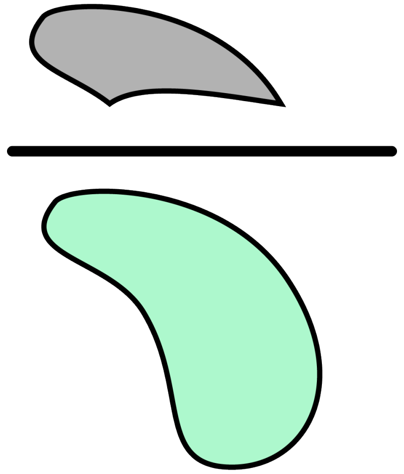
- Mr Black is dead
- We represent the murderer with a random variable `murderer` whose value we dont know. This variable equals either Auburn or Grey.
- $p(murderer = Auburn) = 0.7$
- The "prior" distribution for `murder` is the Bernoulli: $murderer \sim Bernoulli(0.7)$

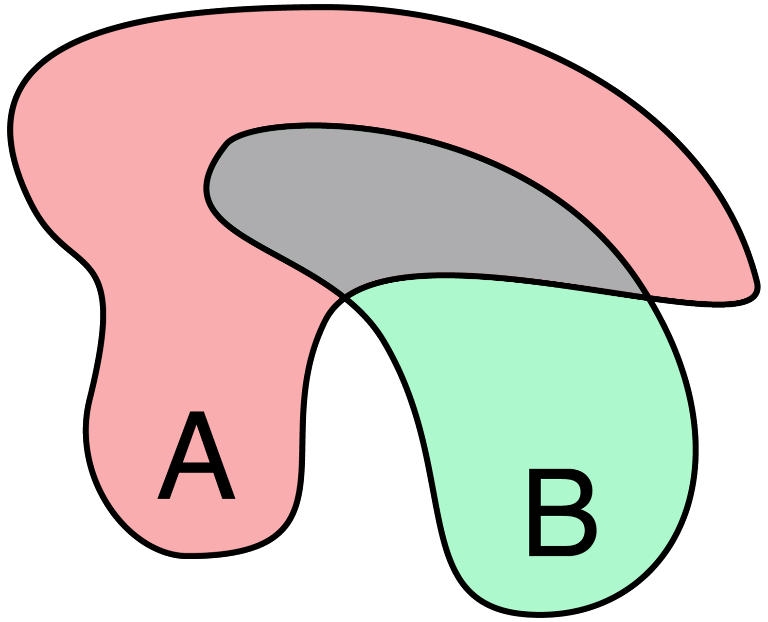
Evidence and conditional probability

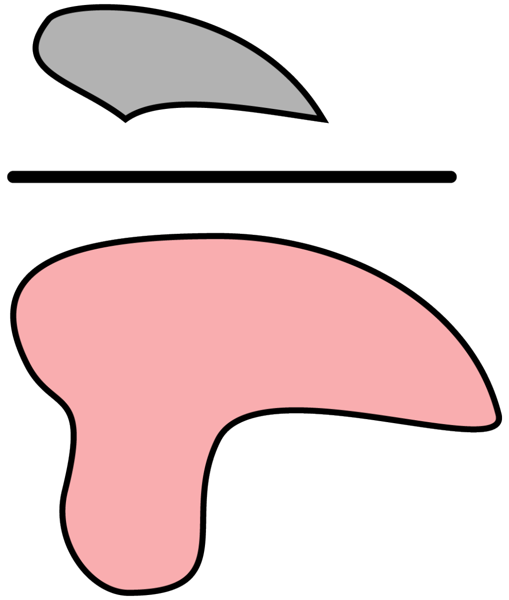
- an ornate ceremonial dagger and an old army revolver are found. We thus introduce a new random variable `weapon`, in addition to the existing random variable `murderer`.

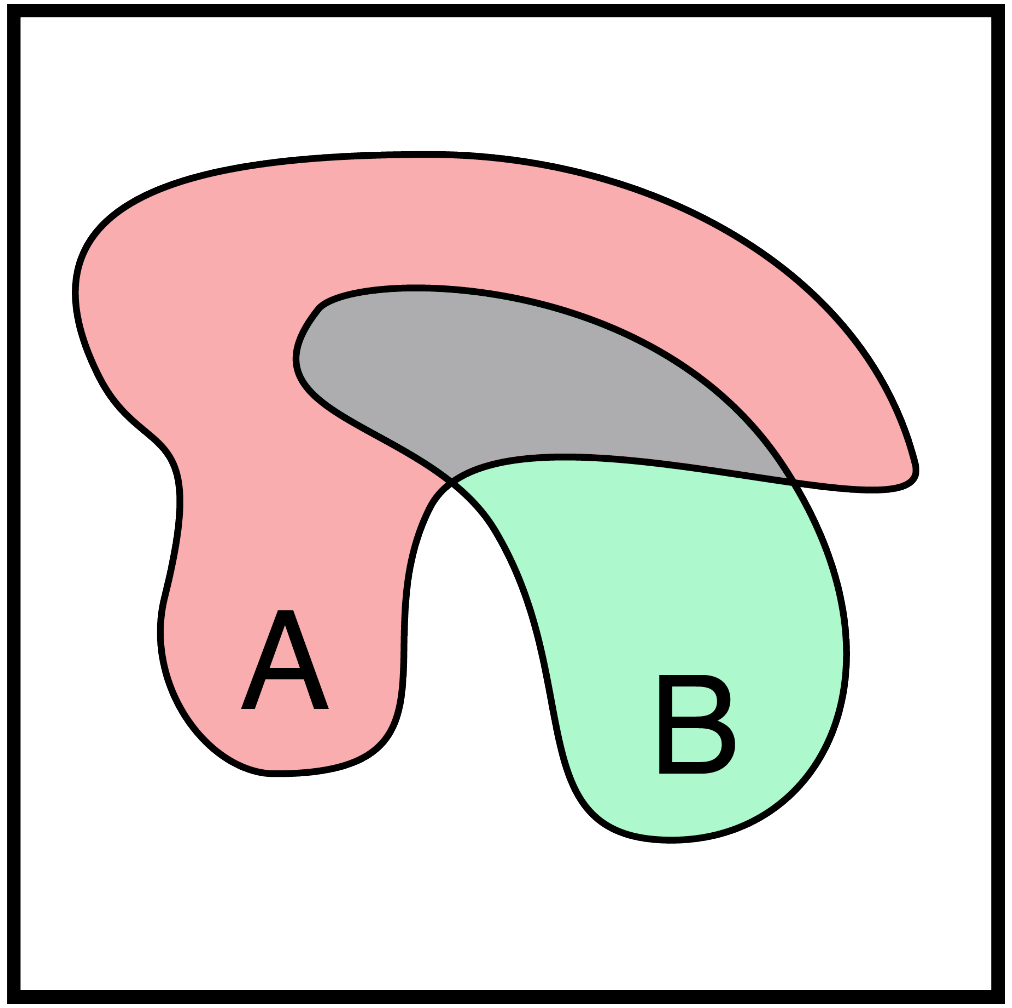
- $p(weapon = revolver \mid murderer = grey) = 0.9$,
  $p(weapon = revolver \mid murderer = auburn) = 0.2$

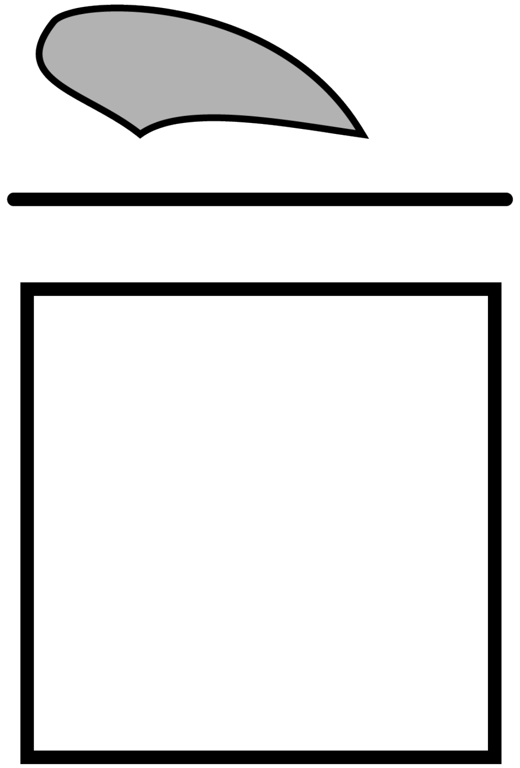


$P(A|B) =$ 

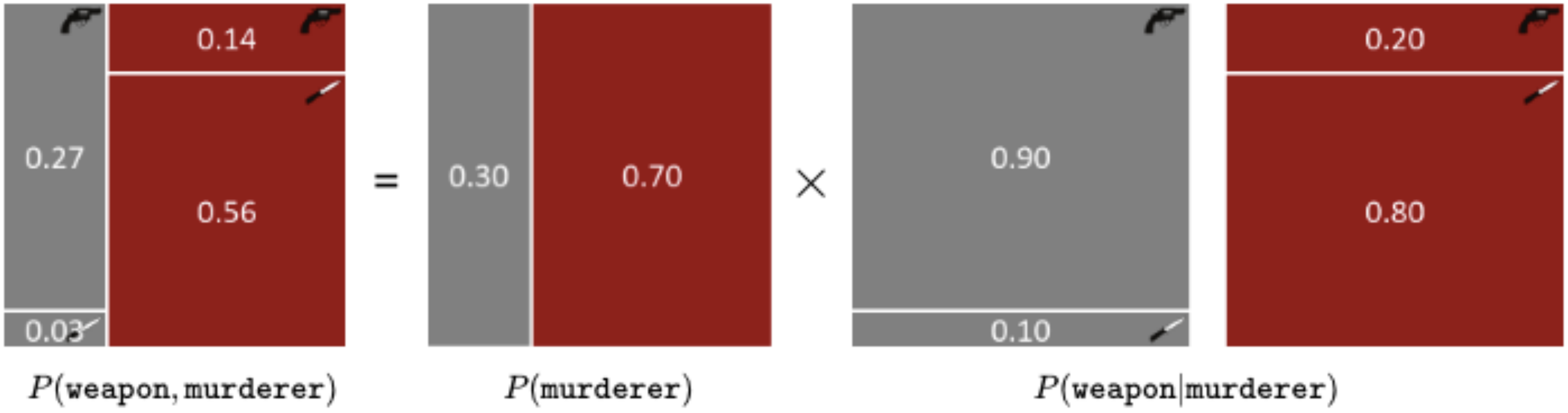


$P(B|A) =$ 



$P(A,B) =$ 

The joint Probability distribution



A probabilistic model is:

- A set of random variables,
- A joint probability distribution over these variables (i.e. a distribution that assigns a probability to every configuration of these variables such that the probabilities add up to 1 over all possible configurations).

Now we condition on some random variables and learn the values of others.

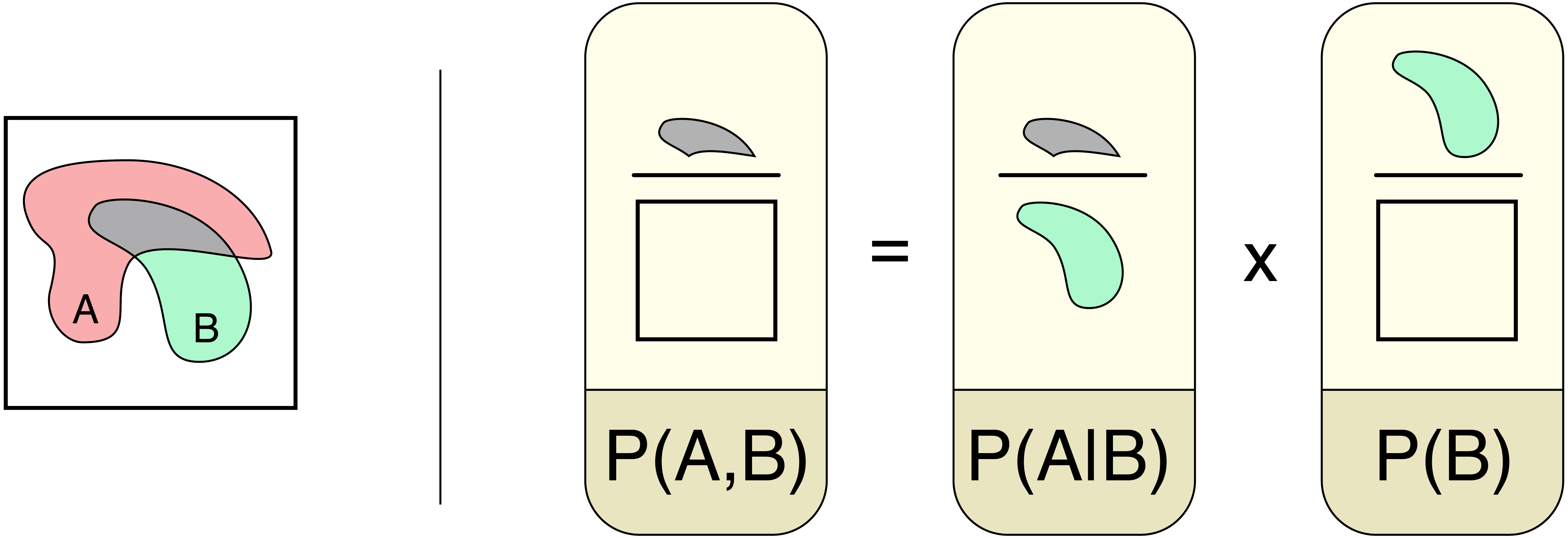
(paraphrased from Model Based Machine Learning)

Rules

1. $P(A, B) = P(A \mid B)P(B)$
2. $P(A) = \sum_B P(A, B) = \sum_B P(A \mid B)P(B)$

$P(A)$ is called the **marginal** distribution of A, obtained by summing or marginalizing over B.

Conditional Rule



Marginal Rule

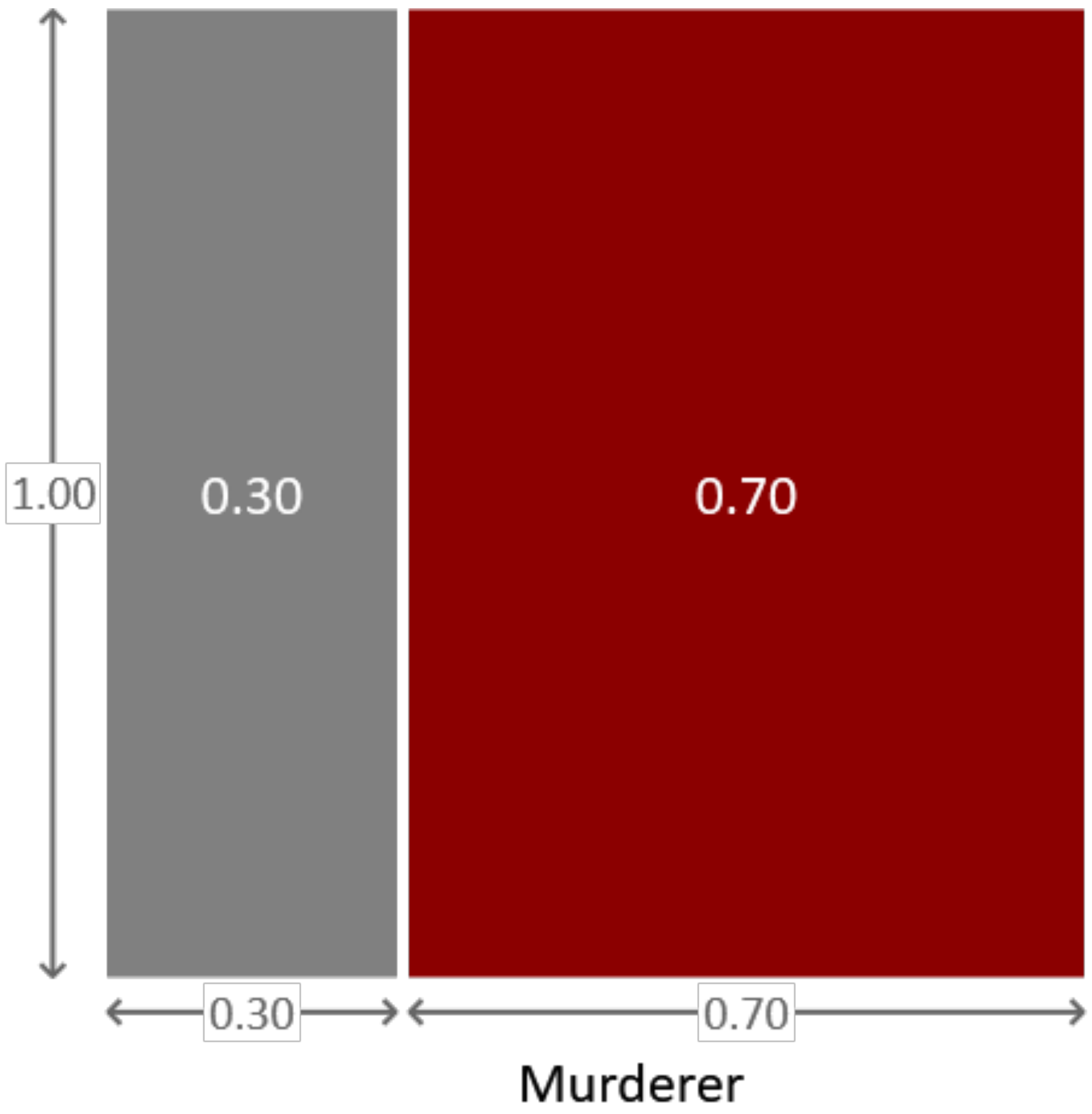
	Vanilla	Chocolate	
Cone	40	60	P(Cone) = 100/150 ≈ 0.66
Cup	20	30	P(Cup) = 50/150 ≈ 0.33
	P(Vanilla) = 60/150 = 0.4	P(Chocolate) = 90/150 = 0.6	

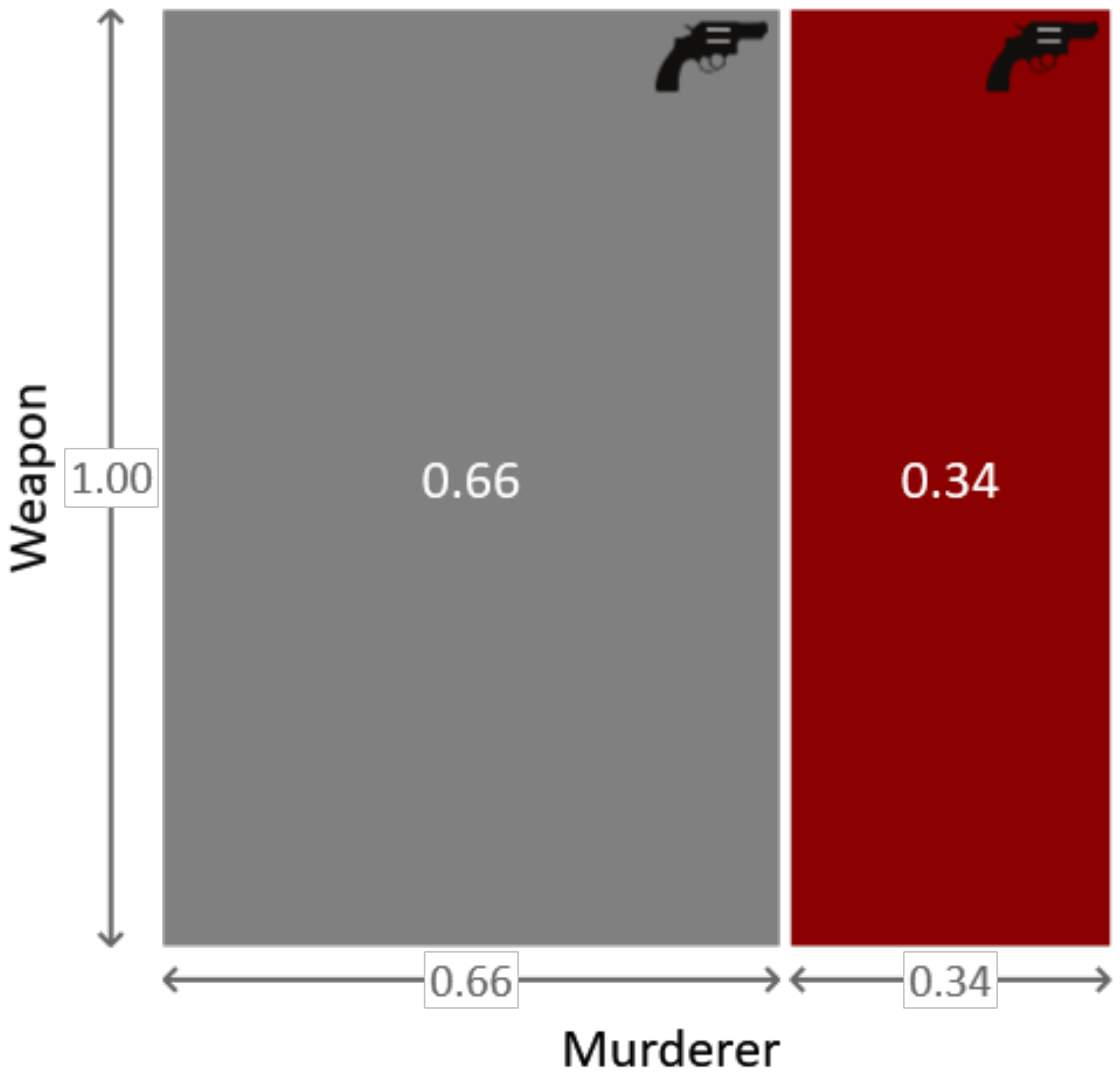
Observation and Inference

- Dr Bayes spots a bullet lodged in the book case.

The process of computing revised probability distributions after we have observed the values of some the random variables, is called inference.

- a principled way from prior to posterior


1.00
0.30
0.70
0.30
0.70
Murderer


Weapon
1.00
0.66
0.34
0.66
0.34
Murderer

Bayes Theorem: Inference without computing the joint distribution

Why? The joint can be computationally hard. Sometimes there are two many "factors"

$$p(y \mid x) = \frac{p(x \mid y)\,p(y)}{p(x)} = \frac{p(x \mid y)\,p(y)}{\sum_{y'} p(x, y')} = \frac{p(x \mid y)\,p(y)}{\sum_{y'} p(x \mid y')p(y')}$$

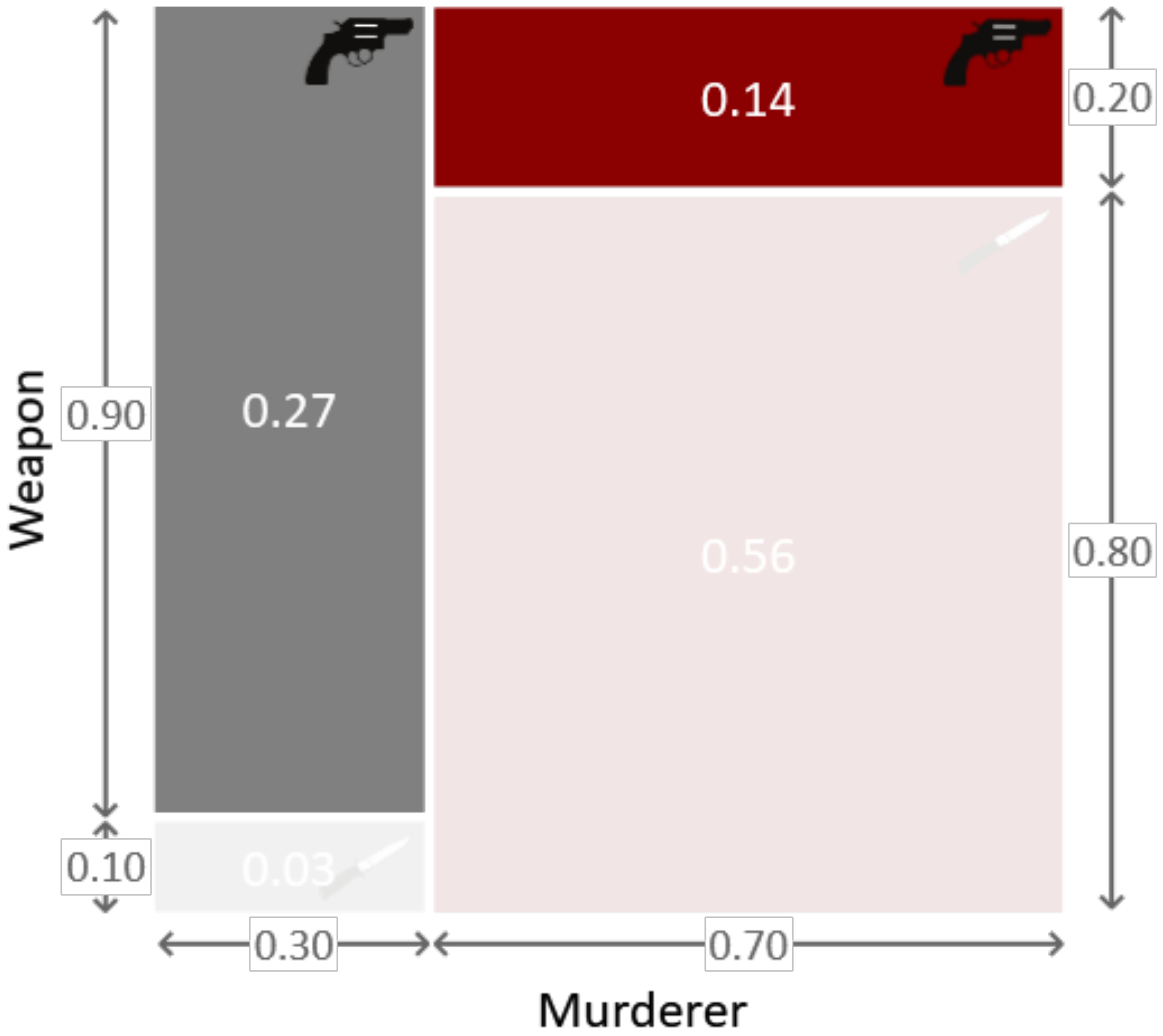
$$P(murderer|weapon) = \frac{P(weapon|murderer)P(murderer)}{P(weapon)}.$$

$$P(weapon) = \sum_{murderer} P(weapon|murderer)P(murderer)$$

$$\text{posterior} = \frac{\text{likelihood} \times \text{prior}}{\text{evidence}}.$$

The `evidence` is just a normalizer and can often be ignored.

The `likelihood function` is NOT a probability distribution over `weapon` (which is known!). It is a function of the random variable `murderer`.



Just ignore the fact that we are in a square!

Lets get precise

Cumulative distribution Function

The **cumulative distribution function**, or the **CDF**, is a function

$$F_X : \mathbb{R} \to [0, 1],$$

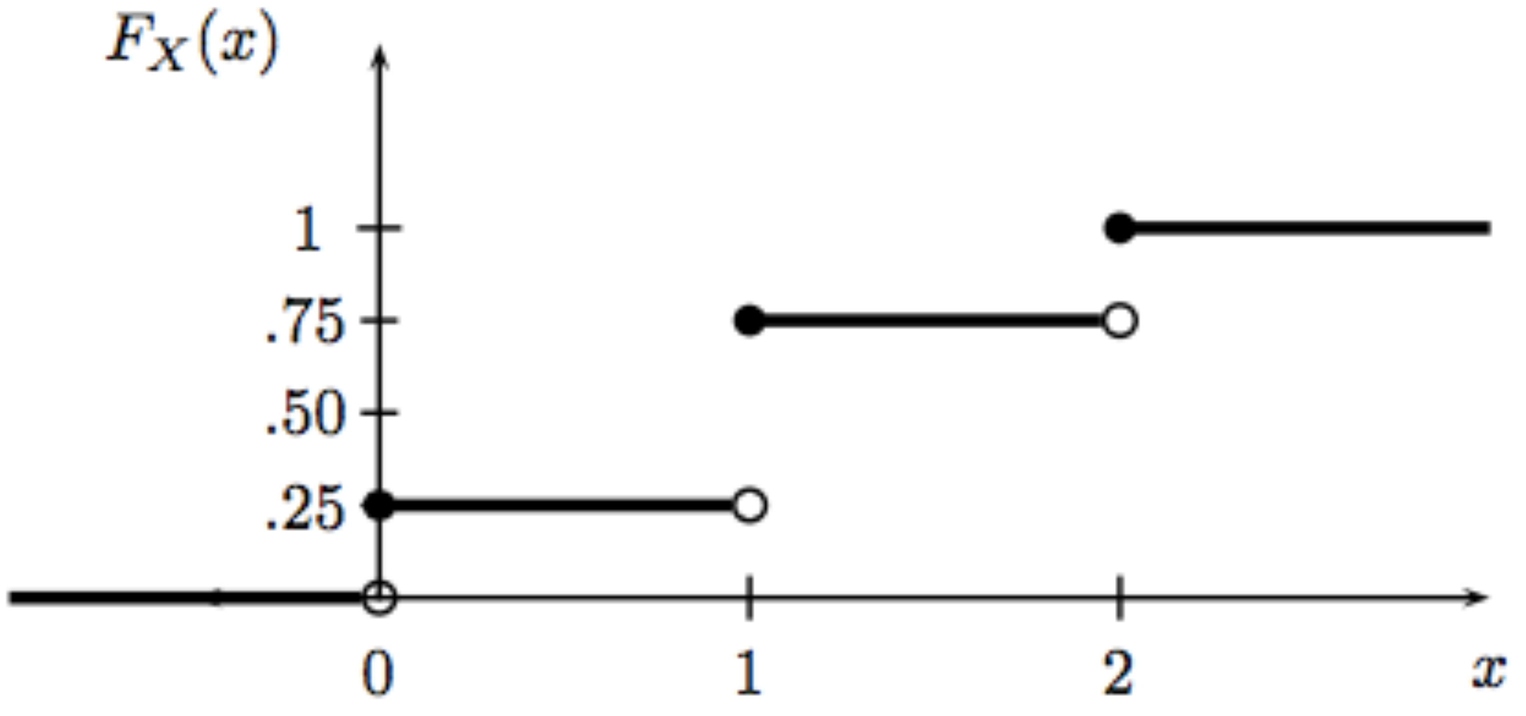
defined by

$$F_X(x) = p(X \leq x).$$

Sometimes also just called *distribution*.

Let X be the random variable representing the number of heads in two coin tosses. Then x = 0, 1 or 2.

CDF:



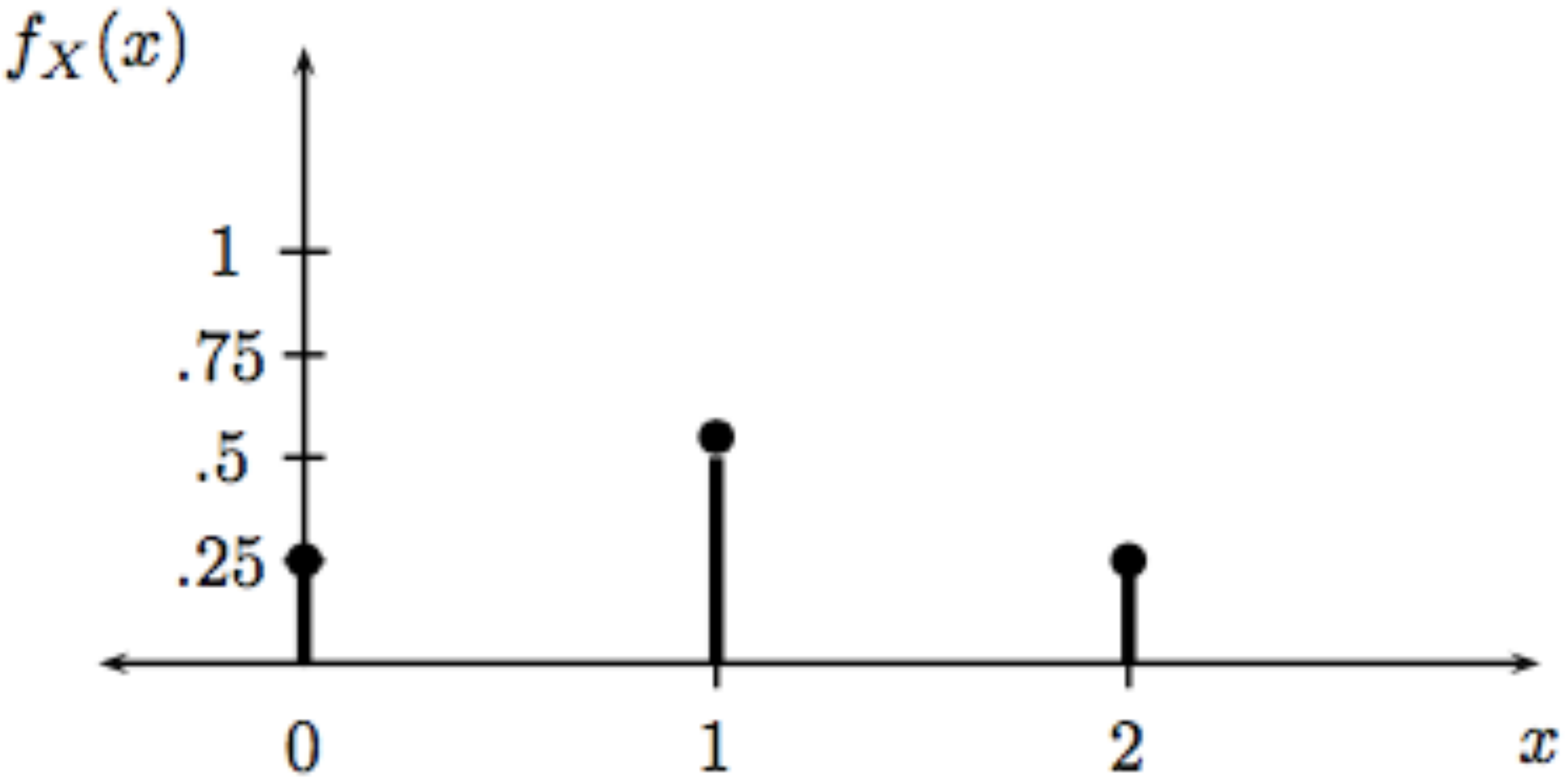
Probability Mass Function

X is called a **discrete random variable** if it takes countably many values $\{x_1, x_2, \ldots\}$.

We define the **probability function** or the **probability mass function** (**pmf**) for X by:

$$f_X(x) = p(X = x)$$

The pmf for the number of heads in two coin tosses:



Probability Density function (pdf)

A random variable is called a **continuous random variable** if there exists a function f_X such that $f_X(x) \geq 0$ for all x, $\int_{-\infty}^{\infty} f_X(x)dx = 1$ and for every a ≤ b,

$$p(a < X < b) = \int_a^b f_X(x)dx$$

Note: $p(X = x) = 0$ for every x. Confusing!

CDF for continuous random variables

$$F_X(x) = \int_{-\infty}^{x} f_X(t)dt$$

and $f_X(x) = \dfrac{dF_X(x)}{dx}$ at all points x at which F_X is differentiable.

Continuous pdfs can be > 1. cdfs bounded in [0,1].

A continuous example: the Uniform(0,1) Distribution

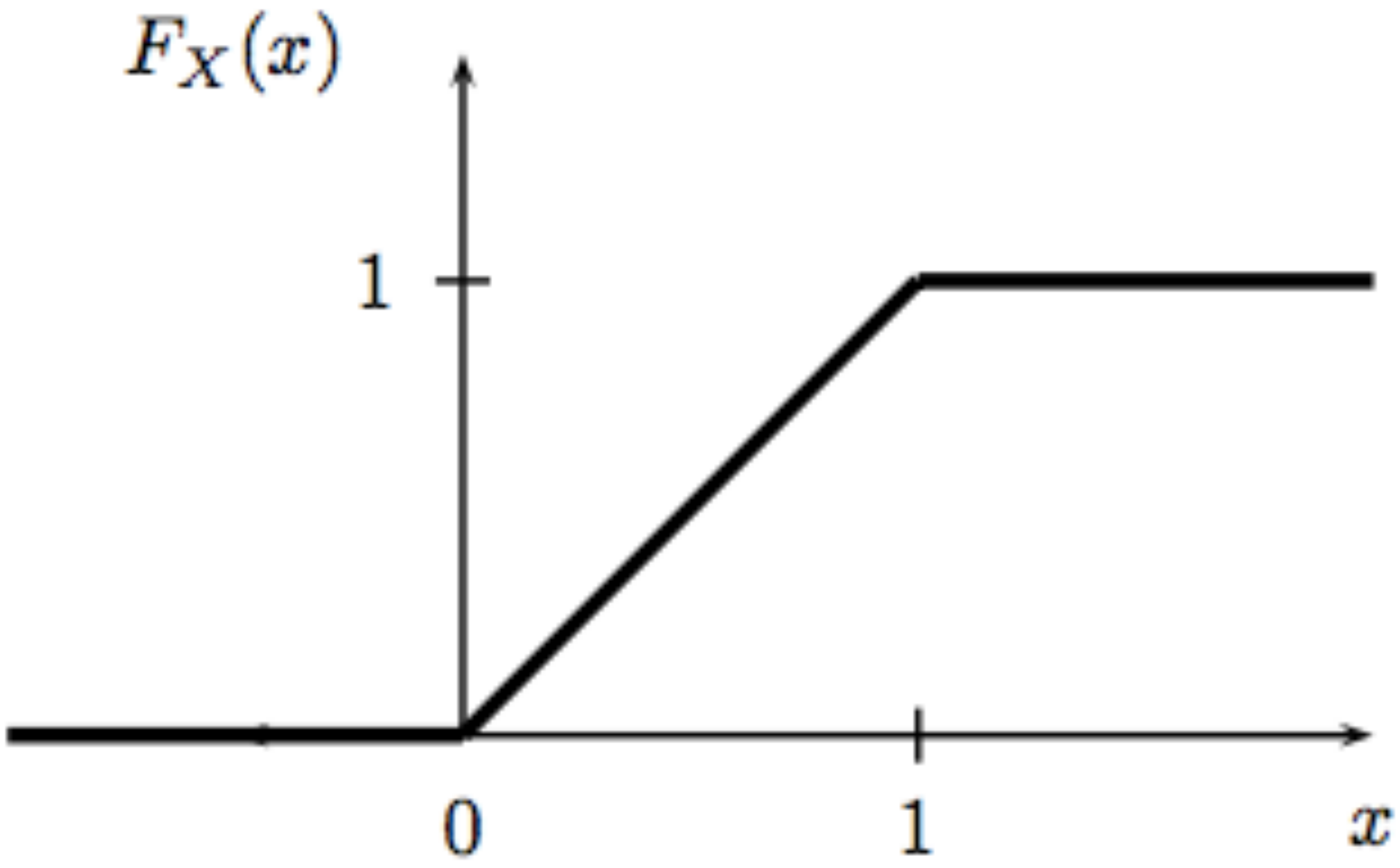
pdf:

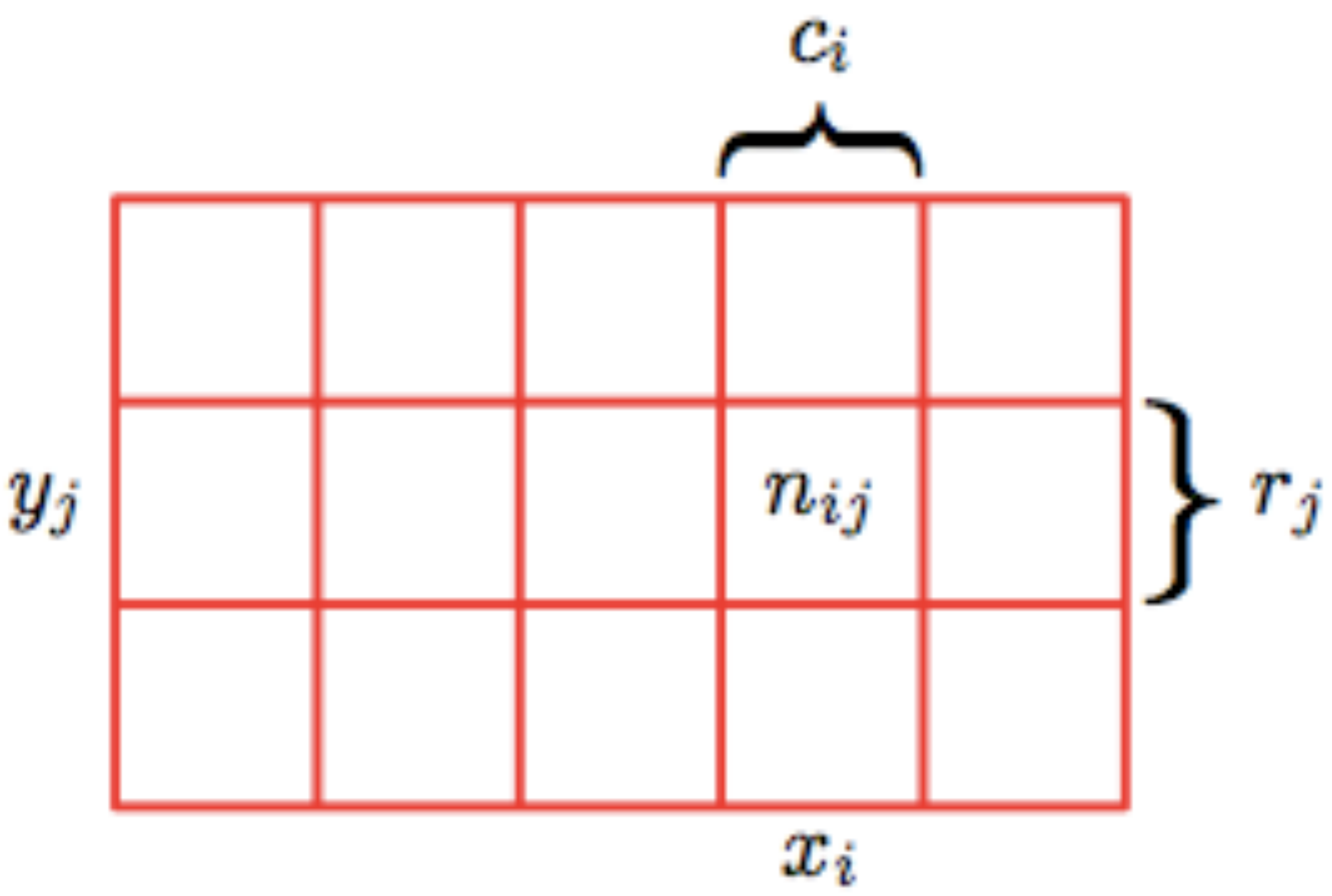
$$f_X(x) = \begin{cases} 1 & \text{for } 0 \leq x \leq 1 \\ 0 & \text{otherwise.} \end{cases}$$

cdf:

$$F_X(x) = \begin{cases} 0 & x \leq 0 \\ x & 0 \leq x \leq 1 \\ 1 & x > 1. \end{cases}$$

cdf:





Marginals and Conditionals

$$p(X = x_i) = \sum_j p(X = x_i, Y = y_j)$$

$$p(Y = y_j \mid X = x_i) \times p(X = x_i) = p(X = x_i, Y = y_j).$$

More generally for hidden variables z:

$$p(x) = \sum_z p(x, z) = \sum_z p(x|z)p(z)$$

Marginals

Marginal mass functions are defined in analog to probabilities:

$$f_X(x) = p(X = x) = \sum_y f(x, y);\;\; f_Y(y) = p(Y = y) = \sum_x f(x, y).$$

Marginal densities are defined using integrals:

$$f_X(x) = \int dy f(x, y);\;\; f_Y(y) = \int dx f(x, y).$$

Conditionals

Conditional mass function is a conditional probability:

$$f_{X|Y}(x \mid y) = p(X = x \mid Y = y) = \frac{p(X = x, Y = y)}{p(Y = y)} = \frac{f_{XY}(x, y)}{f_Y(y)}$$

The same formula holds for densities with some additional requirements $f_Y(y) > 0$ and interpretation:

$$p(X \in A \mid Y = y) = \int_{x \in A} f_{X|Y}(x, y) dx.$$

Bernoulli pmf:

$$f(x) = \begin{cases} 1-p & x = 0 \\ p & x = 1. \end{cases}$$

for p in the range 0 to 1.

$$f(x) = p^x (1-p)^{1-x}$$

for x in the set {0,1}.

What is the cdf?

The big Ideas

create and simulate a data story

perform inference using data story

Data story

- a story of how the data came to be.
- may be a causal story, or a descriptive one (correlational, associative).
- The story must be sufficient to specify an algorithm to simulate new data*.
- a formal **probability model**.

tossing a globe in the air experiment

- toss and catch it. When you catch it, see whats under index finger
- mark `W` for water, `L` for land.
- figure how much of the earth is covered in water
- thus the "data" is the fraction of `W` tosses

Probabilistic Model

1. The true proportion of water is p.

2. Bernoulli probability for each globe toss, where p is thus the probability that you get a `W`. This assumption is one of being **Identically Distributed**.

3. Each globe toss is **Independent** of the other.

Assumptions 2 and 3 taken together are called **IID**, or **Independent and Identially Distributed** Data.

Expectations, LLN, Monte Carlo, and the CLT

- Expectations and some notation
- The Law of large numbers
- Simulation and Monte Carlo for Integration
- Sampling and the CLT
- Errors in Monte Carlo

Expectation $E_f[X]$

Why calculate it?

- we'll see it corresponds to the frequentist notion of probability
- we often want point estimates

Expectations are always with respect to a pmf or density. Often just called the **mean** of the mass function or density. More weight to more probable values.

For the discrete random variable X:

$$E_f[X] = \sum_x x\, f(x).$$

Continuous case:

$$E_f[X] = \int x\, f(x) dx = \int x dF(x),$$

Notation

The expected value, or mean, or first moment, of X is defined to be

$$E_f X = \int x dF(x) = \begin{cases} \sum_x x f(x) & \text{if X is discrete} \\ \int x f(x) dx & \text{if X is continuous} \end{cases}$$

assuming that the sum (or integral) is well defined.

The discrete sum can be said to be an integral with respect to a counting measure.

LOTUS: Law of the unconscious statistician

Also known as **The rule of the lazy statistician**.

Theorem:

if $Y = r(X)$,

$$E[Y] = \int r(x) dF(x)$$

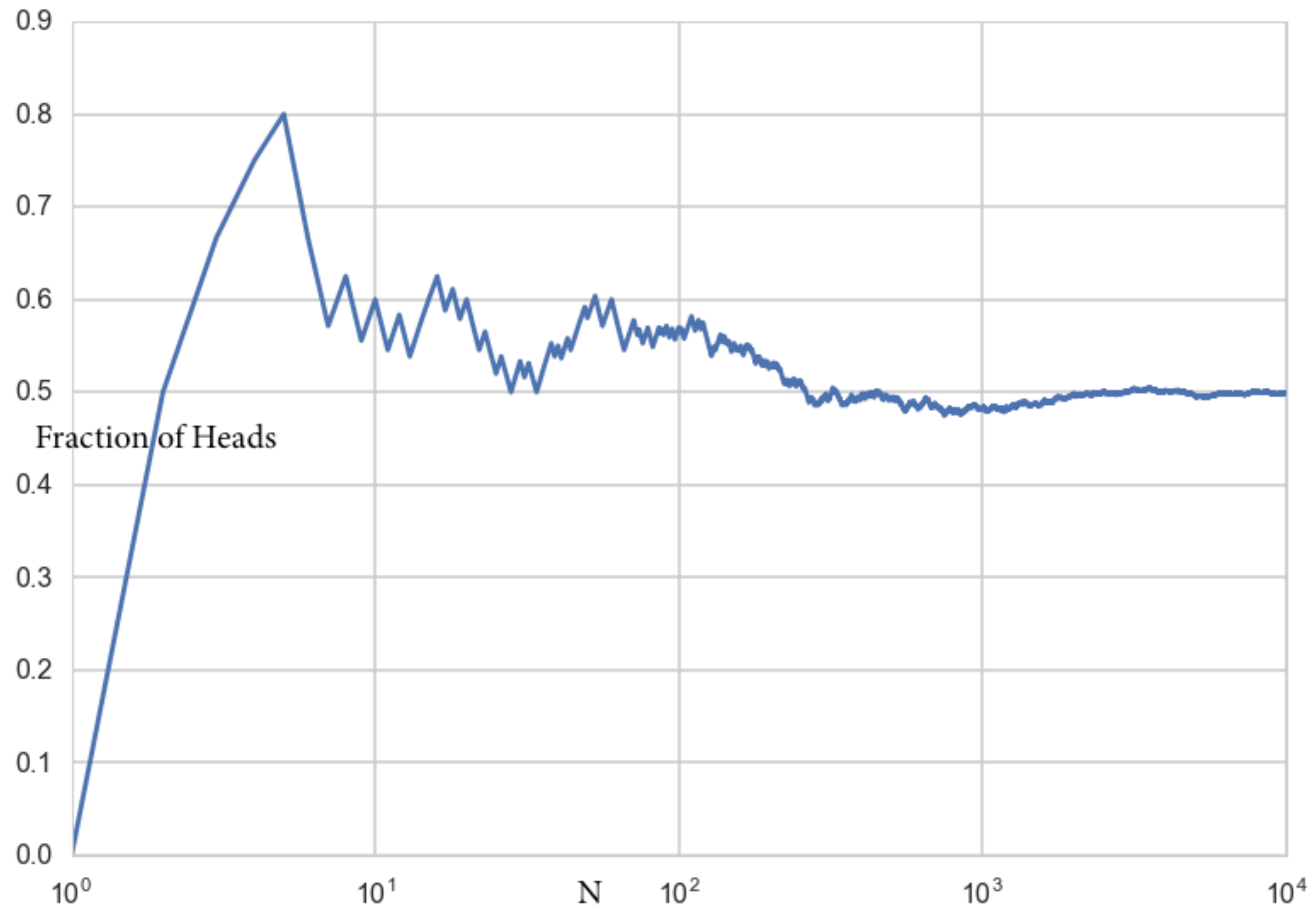
Application: Probability as Expectation

Let A be an event and let $r(x) = I_A(x)$ (Indicator for event A)

Then:

$$E_f[I_A(X)] = \int I_A(x) dF(x) = \int_A f_X(x) dx = p(X \in A)$$

Ever longer sequences for means



Law of Large numbers

Let $x_1, x_2, \ldots, x_n$ be a sequence of IID values from random variable X, which has finite mean μ. Let:

$$S_n = \frac{1}{n}\sum_{i=1}^{n} x_i,$$

Then:

$$S_n \to \mu\, as\, n \to \infty.$$

Frequentist Interpretation of probability

$$E_F[I_A(X)] = p(X \in A)$$

Suppose $Z = I_A(X) \sim Bernoulli(p = P(A))$.

Now if we take a long sequence `seq=10010011100....` from Z, then

$$P(A) = \texttt{mean(seq)} \text{ as } \texttt{length(seq)} \to \infty$$

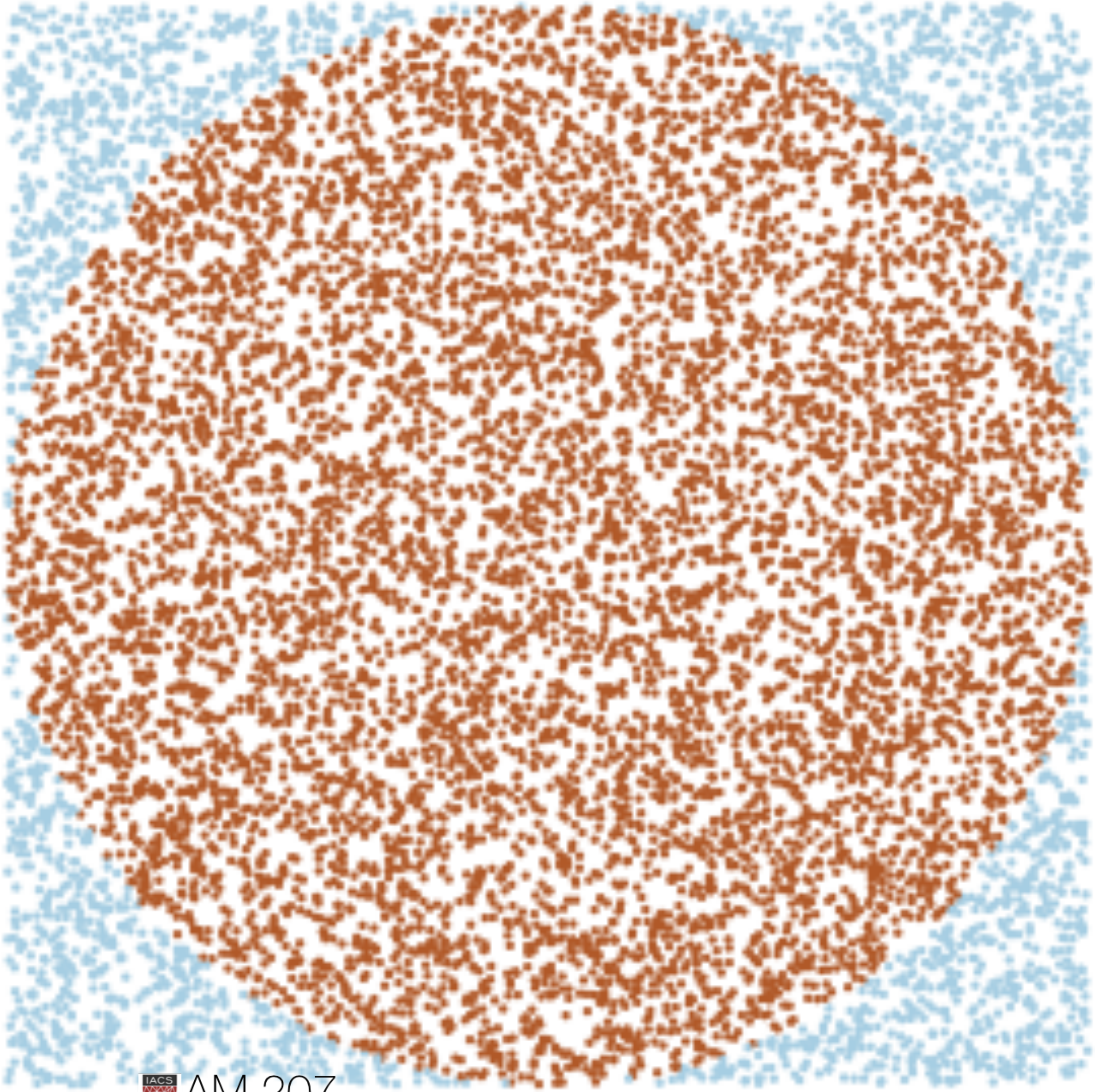
Monte Carlo Algorithm

- use randomness to solve what is often a deterministic problem
- application of the law of large numbers
- integrals, expectations, marginalization
- we'll study optimization, integration, and obtaining draws from a probability distribution

…I wondered whether a more practical method than "abstract thinking" might not be to lay it out say one hundred times and simply observe and count the number of successful plays

...and more generally how to change processes described by certain differential equations into an equivalent form interpretable as a succession of random operations

— Stanislaw Ulam



estimating π

$$A = \int_x \int_y I_{\in C}(x, y) dx dy = \int \int_{\in C} dx dy$$

$$E_f[I_{\in C}(X, Y)] = \int I_{\in C}(X, Y) dF(X, Y)$$

$$= \int \int_{\in C} f_{X,Y}(x, y) dx dy = p(X, Y \in C)$$

If $f_{X,Y}(x, y) \sim Uniform(V)$:

$$= \frac{1}{V} \int \int_{\in C} dx dy = \frac{A}{V}$$

Formalize Monte Carlo Integration idea

For Uniform pdf: $U_{ab}(x) = 1/V = 1/(b-a)$

$$J = \int_a^b f(x) U_{ab}(x)\, dx = \int_a^b f(x)\, dx / V = I/V$$

From LOTUS and the law of large numbers:

$$I = V \times J = V \times E_U[f] = V \times \lim_{n \to \infty} \frac{1}{N} \sum_{x_i \sim U} f(x_i)$$

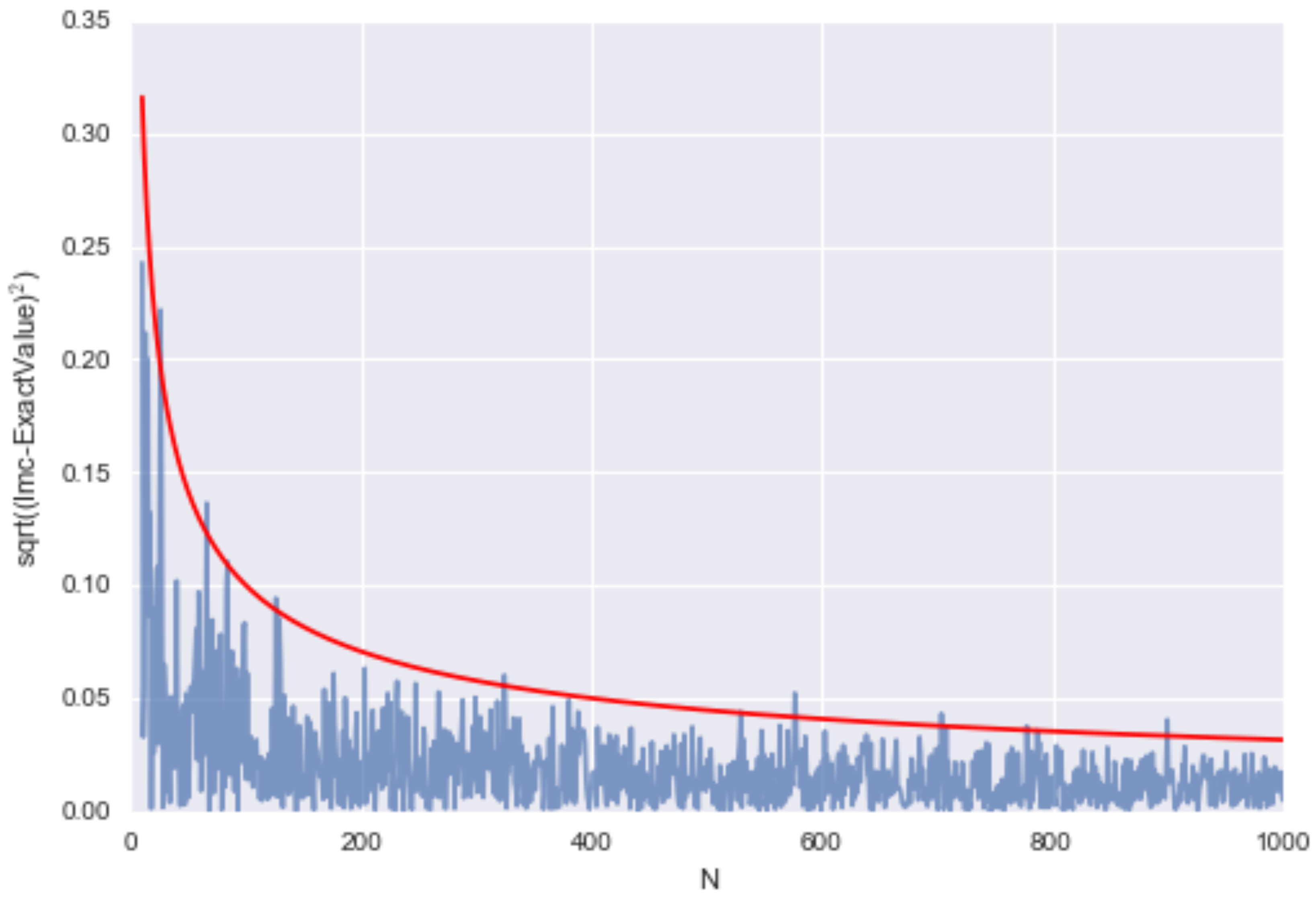
Example

$$I = \int_2^3 \left[x^2 + 4\,x\,\sin(x)\right] dx.$$

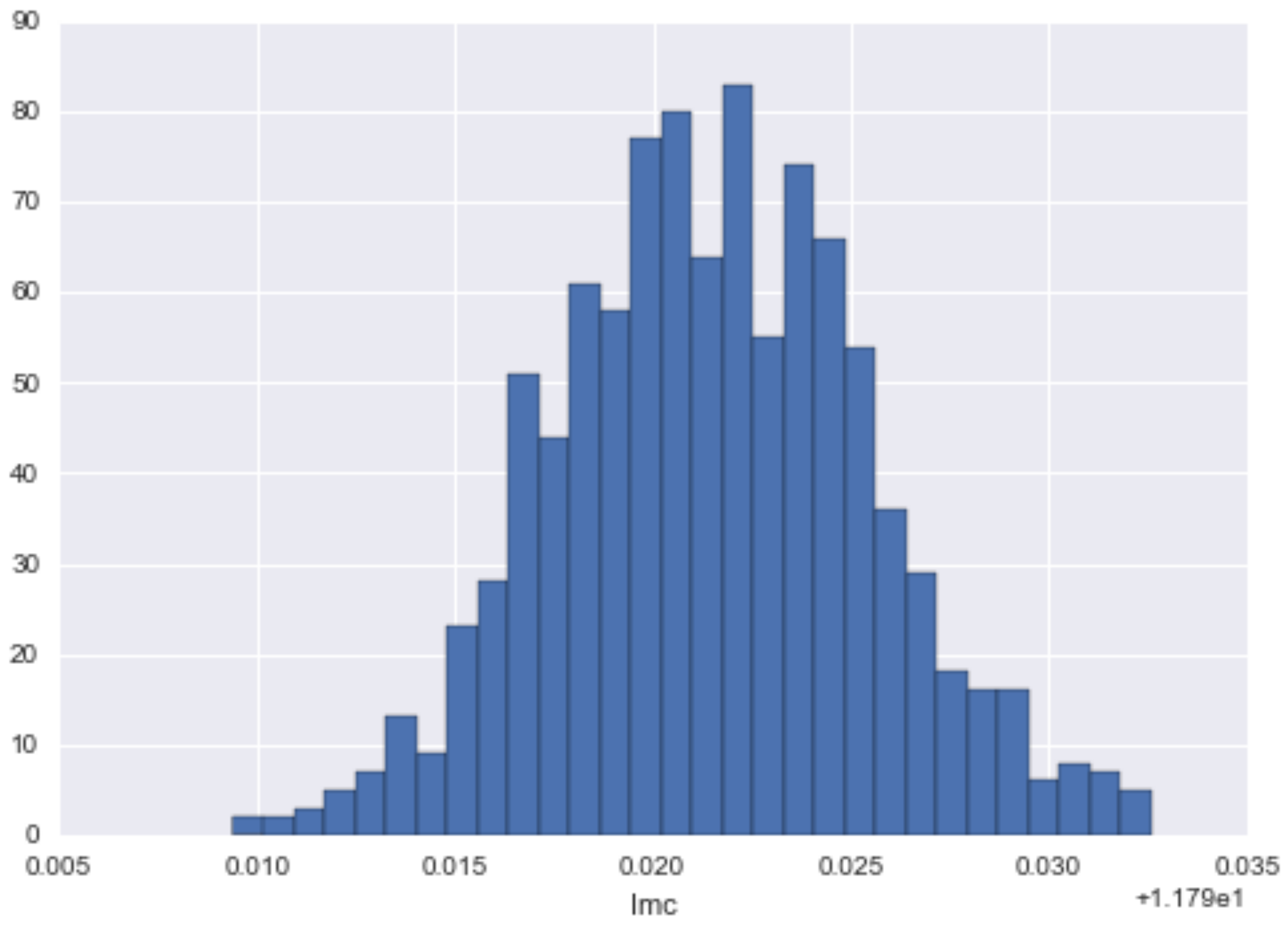
```
def f(x):
    return x**2 + 4*x*np.sin(x)
def intf(x):
    return x**3/3.0+4.0*np.sin(x) - 4.0*x*np.cos(x)
a = 2;
b = 3;
N= 10000
X = np.random.uniform(low=a, high=b, size=N)
Y =f(X)
V = b-a
Imc= V * np.sum(Y)/ N;
exactval=intf(b)-intf(a)
print("Monte Carlo estimation=",Imc, "Exact number=", intf(b)-intf(a))
```

```
Monte Carlo estimation= 11.8120823531 Exact number= 11.8113589251
```

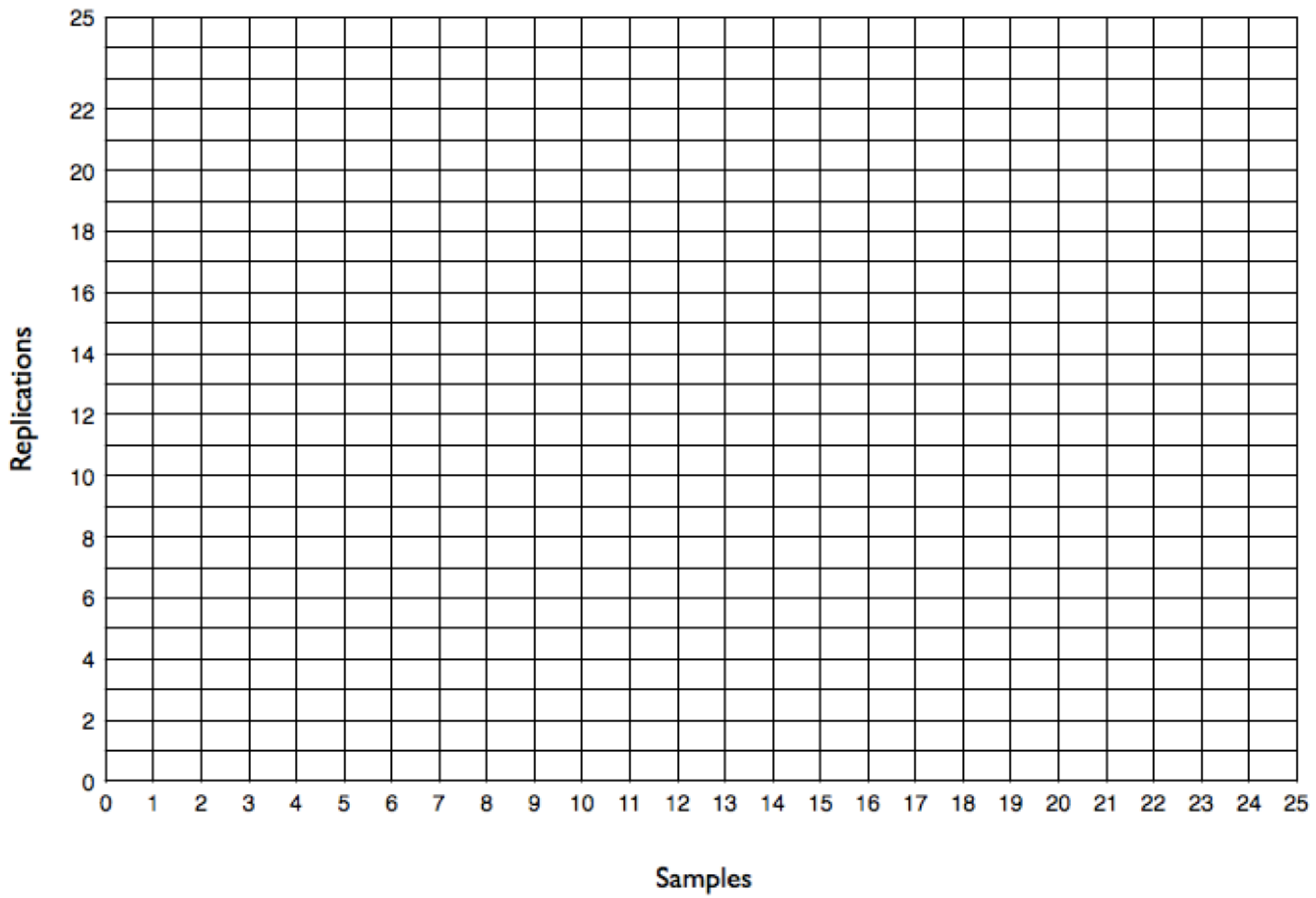
Accuracy as a function of the number of samples



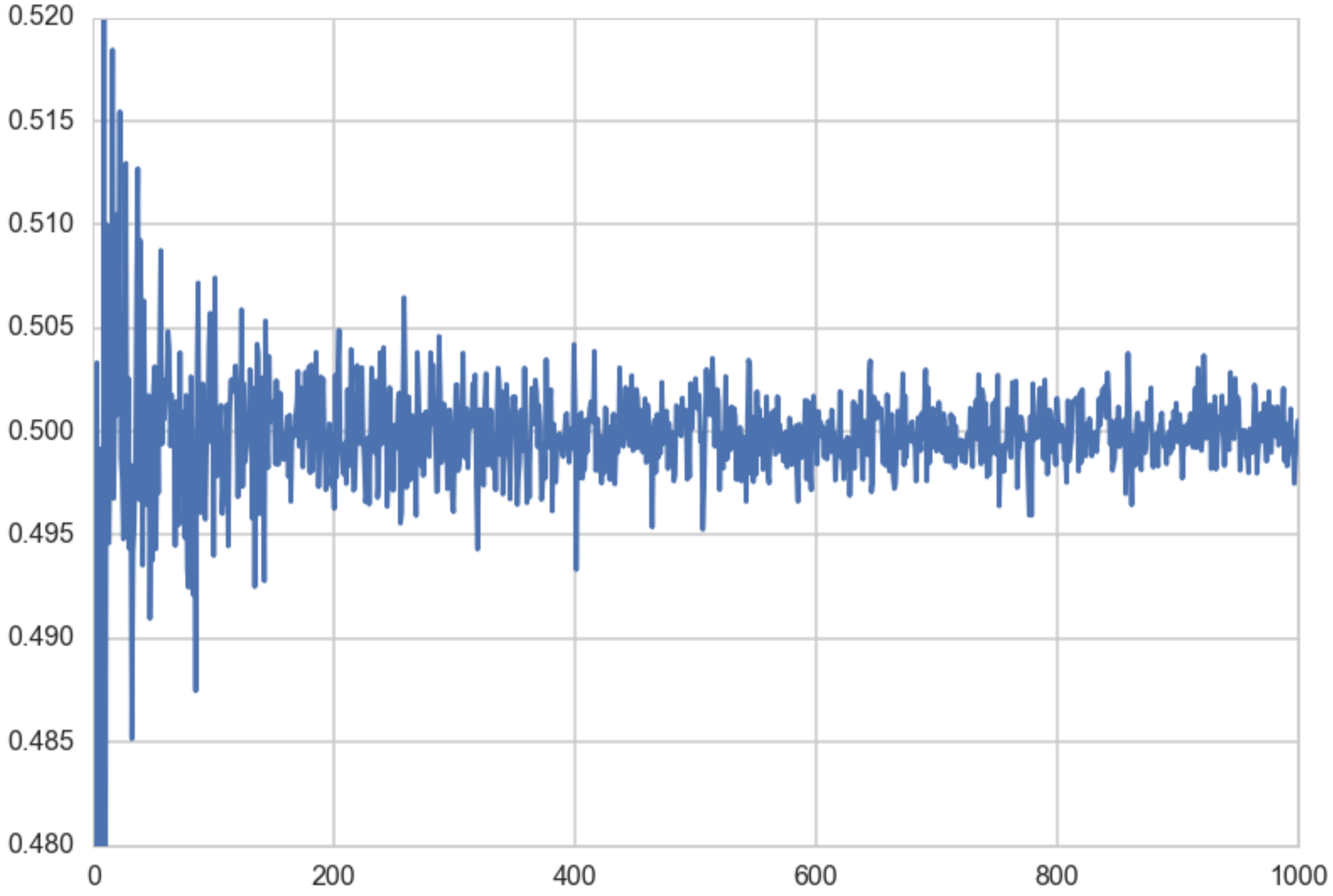
Variance of the estimate



M replications of N coin tosses



sample means: 200 replications of N coin tosses



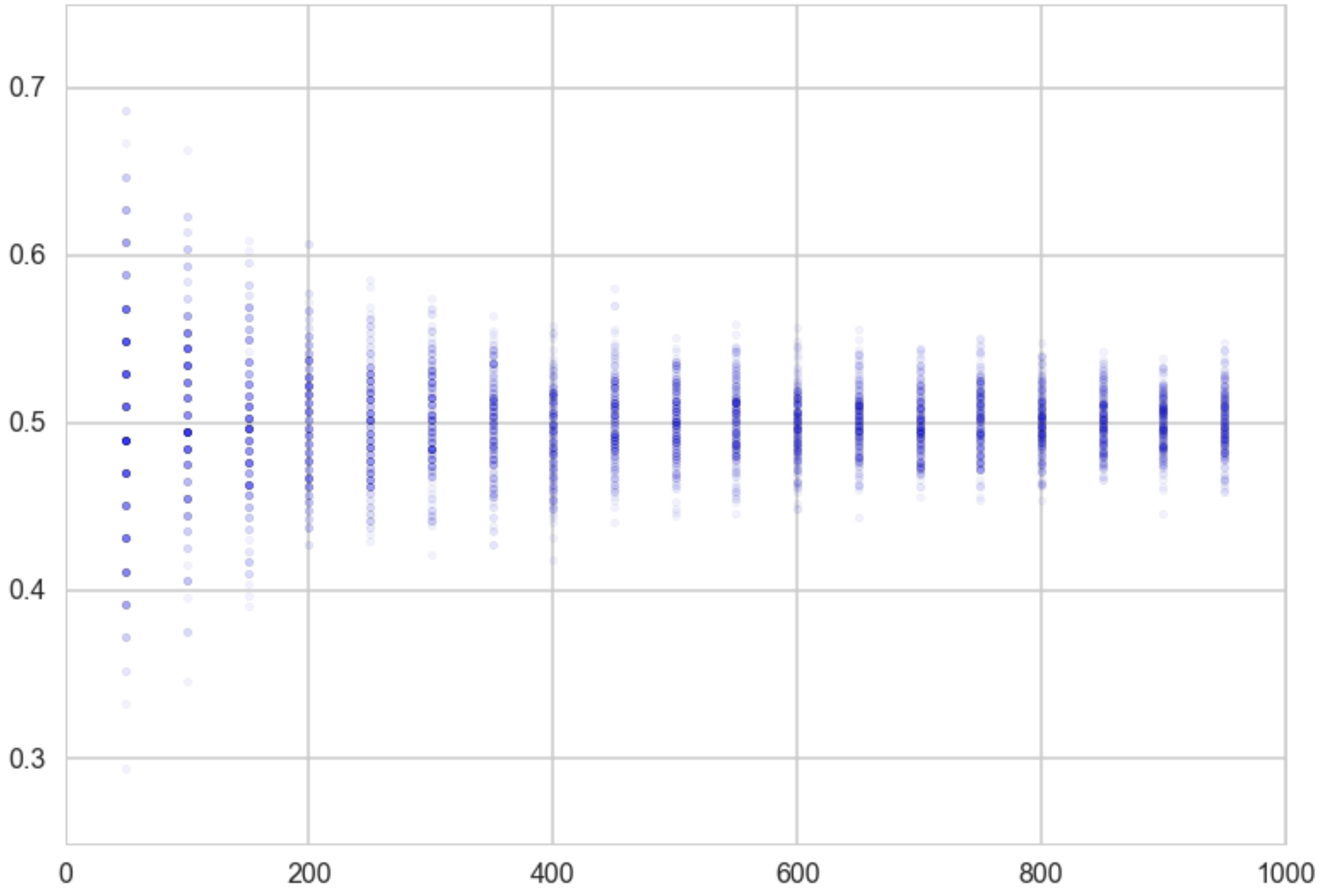
$$E_{\{R\}}(N\,\bar{x}) = E_{\{R\}}(x_1 + x_2 + \ldots + x_N) = E_{\{R\}}(x_1) + E_{\{R\}}(x_2) + \ldots + E_{\{R\}}(x_N)$$

In limit $M \to \infty$ of replications, each of the expectations in RHS can be replaced by the population mean μ using the law of large numbers! Thus:

$$E_{\{R\}}(N\,\bar{x}) = N\,\mu$$
$$E_{\{R\}}(\bar{x}) = \mu$$

In limit $M \to \infty$ of replications the expectation value of the sample means converges to the population mean.

Distribution of Sample Means



Now let underlying distribution have well defined mean μ AND a well defined variance σ^2.

$$V_{\{R\}}(N\,\bar{x}) = V_{\{R\}}(x_1 + x_2 + \ldots + x_N) = V_{\{R\}}(x_1) + V_{\{R\}}(x_2) + \ldots + V_{\{R\}}(x_N)$$

Now in limit $M \to \infty$, each of the variances in the RHS can be replaced by the population variance using the law of large numbers! Thus:

$$V_{\{R\}}(N\,\bar{x}) = N\,\sigma^2$$
$$V(\bar{x}) = \frac{\sigma^2}{N}$$

The Central Limit Theorem (CLT)

Let $x_1, x_2, \ldots, x_n$ be a sequence of IID values from a random variable X. Suppose that X has the finite mean μ AND finite variance σ^2. Then:

$$S_n = \frac{1}{n}\sum_{i=1}^{n} x_i, \text{ converges to}$$

$$S_n \sim N(\mu, \frac{\sigma^2}{n})\, as\, n \rightarrow \infty.$$

Meaning

- weight-watchers' study of 1000 people, average weight is 150 lbs with σ of 30lbs.
- Randomly choose many samples of 100 people each, the mean weights of those samples would cluster around 150lbs with a standard error of 3lbs.
- a different sample of 100 people with an average weight of 170lbs would be more than 6 standard errors beyond the population mean.

Back to Monte Carlo

We want to calculate:

$$S_n(f) = \frac{1}{n}\sum_{i=1}^{n} f(x_i)$$

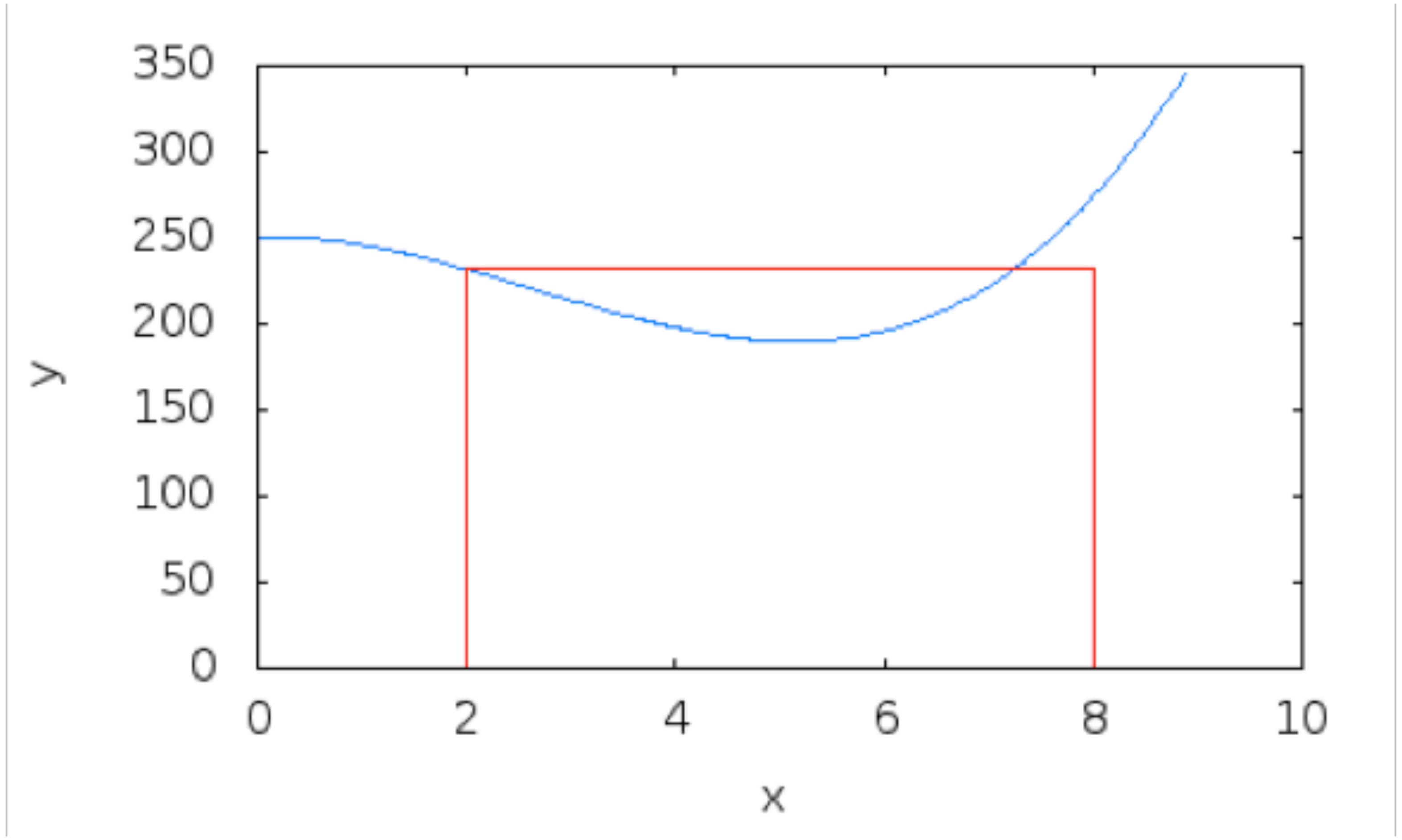
- Whatever $V[f(X)]$ is, the variance of the sampling distribution of the mean goes down as $1/n$
- Thus s goes down as $1/\sqrt{n}$

Why is this important?

- In higher dimensions d, the CLT still holds and the error still scales as $\dfrac{1}{\sqrt{n}}$.
- How does this compete with numerical integration? For $n = N^{1/d}$:
  - left or right rule: $\propto 1/n$, Midpoint rule: $\propto 1/n^2$
  - Trapezoid: $\propto 1/n^2$, Simpson: $\propto 1/n^4$

Basic Numerical Integration idea

(from wikipedia)

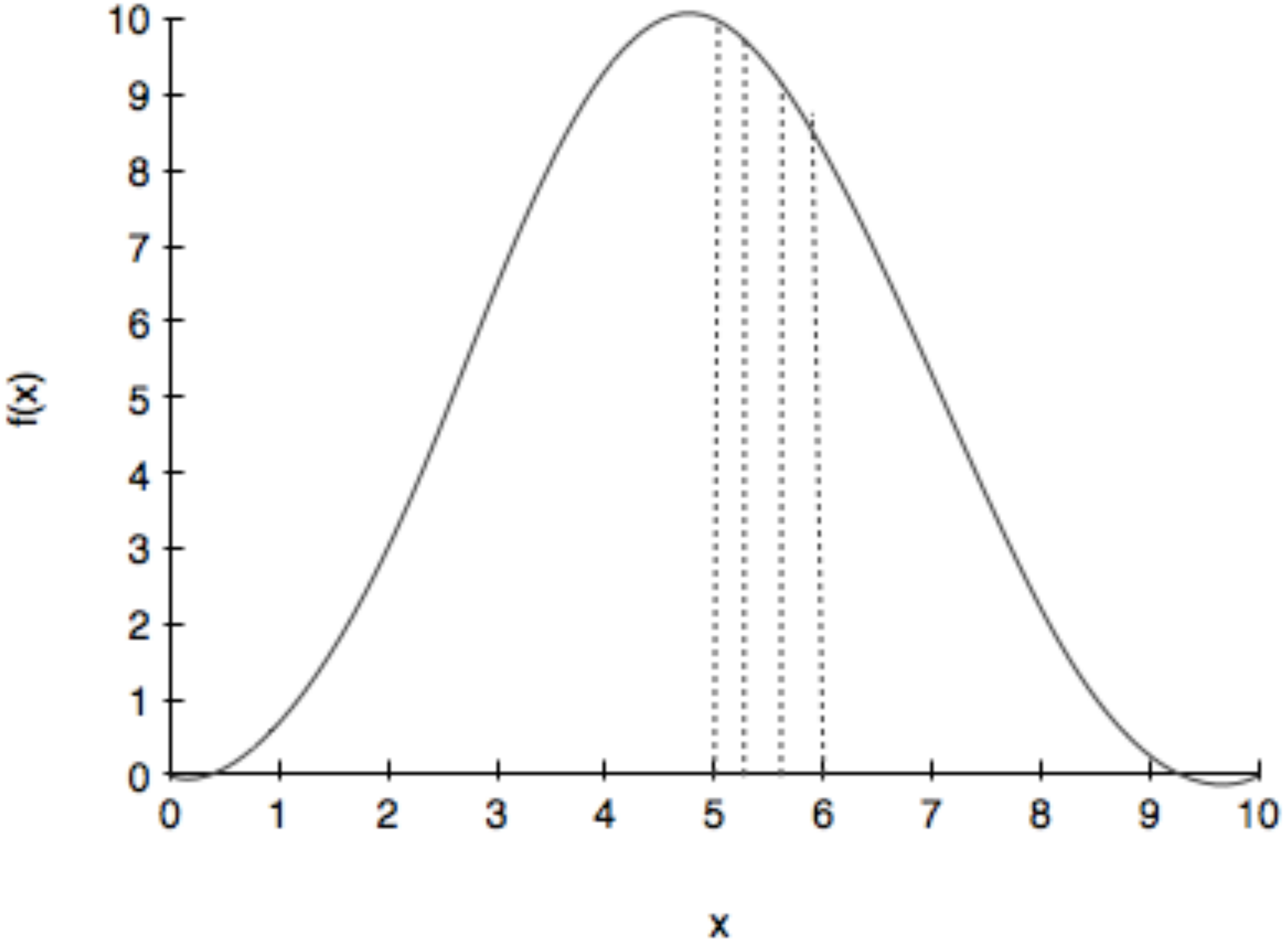


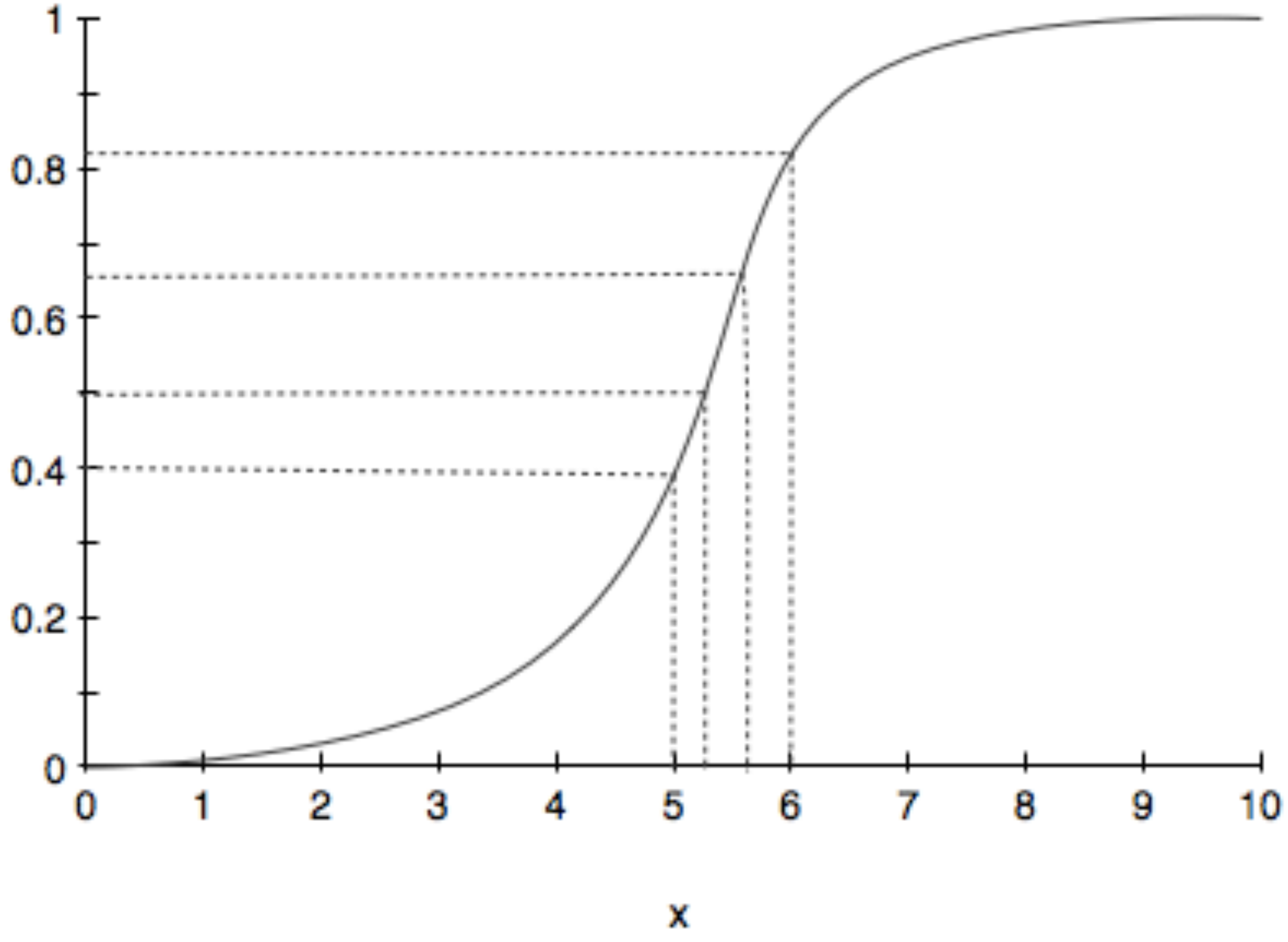
Soon

In order to calculate expectations, do integrals, and do statistics, we must learn how to do

SAMPLING

A taste: Inverse transform





algorithm

The CDF F must be invertible!

1. get a uniform sample u from $Unif(0,1)$
2. solve for x yielding a new equation $x = F^{-1}(u)$ where F is the CDF of the distribution we desire.
3. repeat.

Example: exponential

pdf: $f(x) = \frac{1}{\lambda} e^{-x/\lambda}$ for $x \geq 0$ and $f(x) = 0$ otherwise.

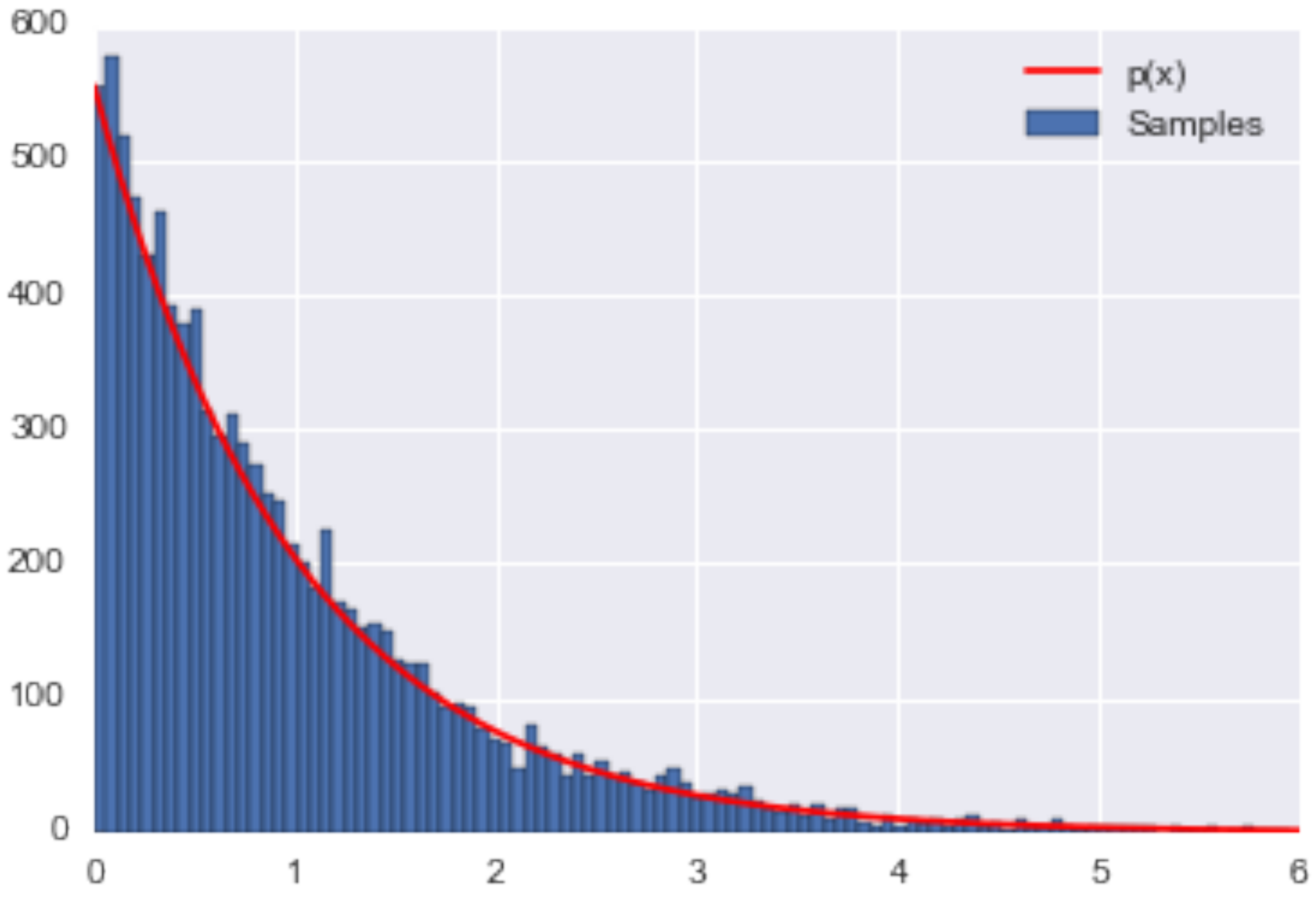
$$u = \int_0^x \frac{1}{\lambda} e^{-x'/\lambda} dx' = 1 - e^{-x/\lambda}$$

Solving for x

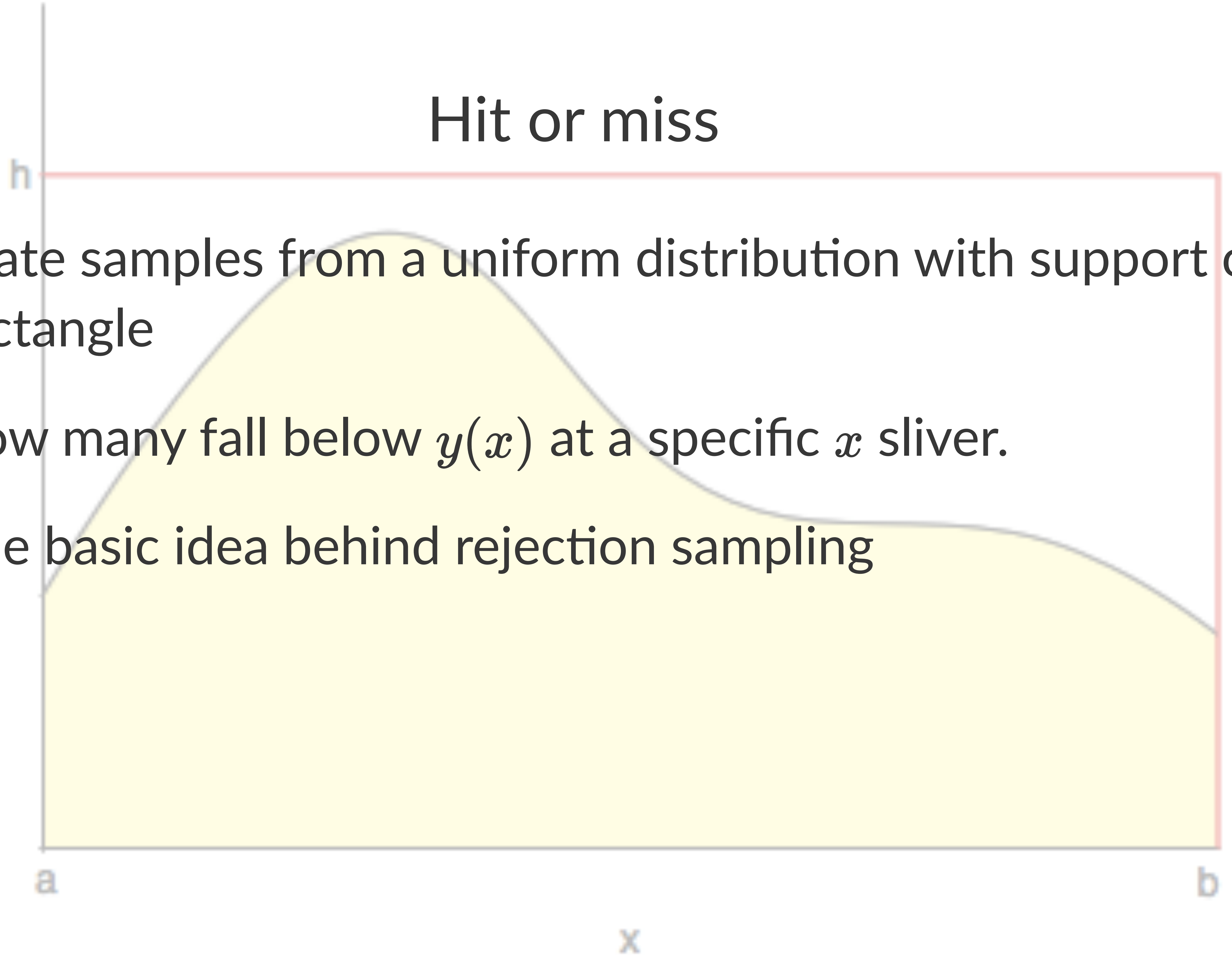
$$x = -\lambda \ln(1 - u)$$

code

```
p = lambda x: np.exp(-x)
CDF = lambda x: 1-np.exp(-x)
invCDF = lambda r: -np.log(1-r) # invert the CDF
xmin = 0 # the lower limit of our domain
xmax = 6 # the upper limit of our domain
rmin = CDF(xmin)
rmax = CDF(xmax)
N = 10000
# generate uniform samples in our range then invert the CDF
# to get samples of our target distribution
R = np.random.uniform(rmin, rmax, N)
X = invCDF(R)
hinfo = np.histogram(X,100)
plt.hist(X,bins=100, label=u'Samples');
# plot our (normalized) function
xvals=np.linspace(xmin, xmax, 1000)
plt.plot(xvals, hinfo[0][0]*p(xvals), 'r', label=u'p(x)')
plt.legend()
```



Hit or miss



- Generate samples from a uniform distribution with support on the rectangle
- See how many fall below $y(x)$ at a specific x sliver.

This is the basic idea behind rejection sampling